MATH 307

HW 4 updated (more hints / fixed typo)

3.5 Non-homogeneous (Undetermined Coeff. Method)

Consider $ay'' + by' + cy = g(t)$

NOT ZERO!

~~Homogeneous :~~

We already know that
if
$$ay_1'' + by_1' + cy_1 = 0 \quad \text{and}$$
$$ay_2'' + by_2' + cy_2 = 0$$
then
$$y = c_1 y_1(t) + c_2 y_2(t) \text{ is also a sol'n to}$$
$$ay'' + by' + cy = 0$$

Nonhomogeneous
If $Y_1(t)$ and $Y_2(t)$ satisfy

$$aY_1'' + bY_1' + cY_1 = g(t)$$
$$aY_2'' + bY_2' + cY_2 = g(t)$$

then
$$a(Y_1'' - Y_2'') + b(Y_1' - Y_2'') + c(Y_1 - Y_2) = 0 \; !$$

HOMOGENEOUS

Thus, if $Y_1(t)$ and $Y_2(t)$ are any sol'ns to $ay'' + by' + cy = g(t)$
then
$$Y_1(t) - Y_2(t) = \underbrace{\text{"a homogeneous sol'n"}}_{c_1 y_1(t) + c_2 y_2(t)}$$

Nonhomogeneous Sol'n Thm

If $y_1(t)$ and $y_2(t)$ are independent sol'ns to $ay''+by'+cy=0$

AND $Y(t)$ is <u>any</u> particular sol'n to $ay''+by'+cy=g(t)$

then all other sol'ns to $ay''+by'+cy=g(t)$ can be written in the form

$$\boxed{y = \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{\text{homogeneous sol'n}} + \underbrace{Y(t)}_{\text{particular sol'n to nonhomogeneous}}}$$

Ex) $y'' + 2y' - 8y = 5e^{3t}$

STEP 1 Homogenous Sol'n

$$r^2 + 2r - 8 = 0$$
$$(r+4)(r-2) = 0$$
$$y_1(t) = e^{-4t}, \quad y_2(t) = e^{2t}$$

STEP 2 Particular Sol'n

"Guess"

$$Y(t) = Ae^{3t}$$
$$Y'(t) = 3Ae^{3t}$$
$$Y''(t) = 9Ae^{3t}$$

METHOD OF UNDETERMINED COEF.

$$(9Ae^{3t}) + 2(3Ae^{3t}) - 8(Ae^{3t}) = 5e^{3t}$$
$$(9A + 6A - 8A)e^{3t} = 5e^{3t}$$
$$7Ae^{3t} = 5e^{3t}$$
$$7A = 5$$
$$A = 5/7 \qquad Y(t) = \frac{5}{7}e^{3t}$$

Step 3 General sol'n

$$\boxed{y(t) = c_1 e^{-4t} + c_2 e^{2t} + \frac{5}{7}e^{3t}}$$

Ex) $y'' + 2y' + y = 3\cos(t)$

$\boxed{\text{STEP 1}}$ $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \quad r = -1$

$y_1(t) = e^{-t}, \quad y_2(t) = te^{-t}$

$\boxed{\text{STEP 2}}$

WRONG GUESS :

$Y(t) = A\cos(t)$

$Y'(t) = -A\sin(t)$

$Y''(t) = -A\cos(t)$

$-A\cos(t) - 2A\sin(t) + A\cos(t) = 3\cos(t)$

$-2A\sin(t) = 3\cos(t)$

???

NO CONSTANT

RIGHT GUESS :

$Y(t) = A\cos(t) + B\sin(t)$

$Y'(t) = -A\sin(t) + B\cos(t)$

$Y''(t) = -A\cos(t) - B\sin(t)$

$-A\cos(t) - B\sin(t) - 2A\sin(t) + 2B\cos(t) + A\cos(t) + B\sin(t) = 3\cos(t)$

$$2B\cos(t) - 2A\sin(t) = 3\cos(t)$$

$2B = 3 \qquad B = 3/2$

$-2A = 0 \qquad A = 0$

$$Y(t) = 3/2 \sin(t)$$

$\boxed{\text{Step 3}}$ $\boxed{y(t) = c_1 e^{-t} + c_2 te^{-t} + \frac{3}{2}\sin(t)}$

METHOD OF UNDETERMINED COEFFICIENTS

To find a particular sol'n to

$$ay'' + by' + cy = g(t)$$

GUESS THE FOLLOWING FORMS

$g(t) = e^{rt} \Rightarrow Y(t) = Ae^{rt}$

$g(t) = \cos(\omega t)$ or $\sin(\omega t) \Rightarrow Y(t) = A\cos(\omega t) + B\sin(\omega t)$

$g(t) = \alpha t + \beta \Rightarrow Y(t) = At + B$

$g(t) = t^2 \Rightarrow Y(t) = At^2 + Bt + C$

For sums,

$g(t) = t + e^{5t} \Rightarrow Y(t) = At + B + Ce^{5t}$

For products

$g(t) = t^2 e^{3t} \Rightarrow Y(t) = (At^2 + Bt + C)e^{3t}$

Ex) $y'' + 3y = 2t + 1$

STEP 1 $r^2 + 3 = 0 \Rightarrow r = \pm\sqrt{3}\, i \qquad \lambda = 0 \quad \omega = \sqrt{3}$

$y_1(t) = \cos(\sqrt{3}t) \qquad y_2(t) = \sin(\sqrt{3}t)$

STEP 2 "GUESS" $Y(t) = At + B$

$Y'(t) = A$

$Y''(t) = 0$

$0 + 3(At + B) = 2t + 1$

$3At + 3B = 2t + 1$

$3A = 2 \qquad A = 2/3$

$3B = 1 \qquad B = 1/3$

Step 3 $\boxed{y(t) = c_1\cos(\sqrt{3}t) + c_2\sin(\sqrt{3}t) + \frac{2}{3}t + \frac{1}{3}}$

Ex) $y'' - 4y = e^{2t}$

$\boxed{\text{STEP 1}}$ $r^2 - 4 = 0$ $\quad r = \pm 2$

$y_1(t) = e^{2t}$ $\quad , y_2(t) = e^{-2t}$

$\boxed{\text{STEP 2}}$

WRONG GUESS:

$Y(t) = Ae^{2t}$

$Y'(t) = 2Ae^{2t}$

$Y''(t) = 4Ae^{2t}$

$y'' - 4y \stackrel{?}{=} e^{2t}$

$4Ae^{2t} - 4Ae^{2t} \stackrel{?}{=} e^{2t}$

$0 = e^{2t}$

???

$\downarrow$

STOP

METHOD OF REDUCTION

$Y(t) = v(t)\, e^{2t}$

$\Downarrow$

$v(t) = t$

RIGHT GUESS:

$Y(t) = Ate^{2t}$

$Y'(t) = Ae^{2t} + 2Ate^{2t} = (A + 2At)e^{2t}$

$Y''(t) = 2Ae^{2t} + 2(A + 2At)e^{2t}$

$= (4A + 4At)e^{2t}$

$y'' - 4y = e^{2t}$

$(4A + 4At)e^{2t} - 4Ate^{2t} = e^{2t}$

$4Ae^{2t} = e^{2t}$

$4A = 1$

$\boxed{A = 1/4}$

$\boxed{\text{Step 3}}$ $\boxed{y = c_1 e^{2t} + c_2 e^{-2t} + \tfrac{1}{4} te^{2t}}$

CONCLUSION: If your initial guess is already a constant multiple of a homogeneous sol'n, then multiply by t.

Ex) $y'' + 3y' - 4y = 4t - 5e^{-4t}$ $\quad y(0) = 1, y'(0) = 0$

$\boxed{\text{STEP 1}}$ $r^2 + 3r - 4 = 0$

$(r+4)(r-1) = 0$

$y_1(t) = e^{-4t}, \quad y_2(t) = e^{t}$

$\boxed{\text{STEP 2}}$ $Y(t) = At + B + C t e^{-4t}$

$Y'(t) = A + Ce^{-4t} - 4C te^{-4t}$

$Y''(t) = -4Ce^{-4t} - 4Ce^{-4t} + 16Cte^{-4t}$

$= (-8C + 16Ct)e^{-4t}$

$(-8C + 16Ct)e^{-4t} + 3A + 3(C - 4Ct)e^{-4t} - 4At - 4B - 4Cte^{-4t}$

$= 4t - 5e^{-4t}$

$\boxed{\text{IN FRONT OF } t}$: $-4A = 4$ $\quad A = -1$

$\boxed{\text{CONSTANT}}$: $3A - 4B = 0 \Rightarrow -4B = 3$ $\quad B = -3/4$

$\boxed{\text{IN FRONT OF } e^{-4t}}$: $-8C + \cancel{16Ct} + 3C - \cancel{12Ct} - \cancel{4Ct} = -5$

$-5C = -5$

$C = 1$

$\boxed{\text{Step 3}}$ $y(t) = c_1 e^{-4t} + c_2 e^{t} - t - 3/4 + te^{-4t}$

$\boxed{\text{INITIAL CONDITIONS}}$ $y(0) = 1 \Rightarrow$ $c_1 + c_2 - 3/4 = 1$

$y'(t) = -4c_1 e^{-4t} + c_2 e^{t} - 1 + e^{-4t} - 4te^{-4t}$

$y'(0) = 0 \Rightarrow -4c_1 + c_2 - 1 + 1 = 0$

$-4c_1 + c_2 = 0$

$c_1 = 7/5, \quad c_2 = 7/20$

MATH 307

HW 5, HAND OUT, $\boxed{\text{TEST PREP FRIDAY}}$

3.5 (continued)

Recall: Given $ay'' + by' + cy = g(t)$

$\boxed{1}$ Solve $ay'' + by' + cy = 0 \Rightarrow y = y_1(t), y = y_2(t)$

$\boxed{2}$ FIND A PARTICULAR SOL'N

$Y(t) = Ae^{rt}$,

$Y(t) = At^2 + Bt + C$

$Y(t) = A\cos(t) + B\sin(t)$

$\boxed{3}$ $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$

$\boxed{\text{ENTRY TASK}}$ What is the form of a particular sol'n?

$\boxed{1}$ $y'' - y = \cos(4t)$ $\boxed{Y(t) = A\cos(4t) + B\sin(4t)}$

$r^2 - 1 = 0$ $r = \pm 1$ $y_1(t) = e^{-t}, y_2(t) = e^t$

$\boxed{2}$ $y'' - 9y = t^2 + 7t + e^{3t}$ $\boxed{Y(t) = At^2 + Bt + C + Dte^{3t}}$

$r^2 - 9 = 0$, $r = \pm 9$, $y_1(t) = e^{-3t}, y_2(t) = e^{3t}$

why

ASIDE: $Y(t) = u(t)e^{3t}$ reduction of order

$\Rightarrow u(t) = t$

For $\cos(wt)$, $\sin(wt)$, e^{rt}, polynomials

- If $g(t)$ is a homogeneous sol'n, then multiply the form by t.

$\boxed{3}$ $y'' - 10y' + 25y = 2e^{5t} + te^t\cos(3t)$

$r^2 - 10r + 25 = 0$

$(r-5)^2 = 0$ $y_1(t) = e^{5t}$ $y_2(t) = te^{5t}$

$$\boxed{Y(t) = At^2e^{5t} + (Bt + C)e^t(D\cos(3t) + E\sin(3t))}$$

Ex $y'' + 3y' - 4y = 4t - 5e^{-4t}$

STEP 1 $r^2 + 3r - 4 = (r+4)(r-1) = 0$

$y_1(t) = e^{-4t}, \quad y_2(t) = e^{t}$

STEP 2 $Y(t) = At + B + Cte^{-4t}$

$Y'(t) = A + Ce^{-4t} - 4Cte^{-4t}$

$Y''(t) = -4Ce^{-4t} - 4Ce^{-4t} + 16Cte^{-4t}$

$$\overbrace{-8Ce^{-4t} + 16\cancel{Cte^{-4t}}}^{y''} + \overbrace{3A + 3Ce^{-4t} - 12\cancel{Cte^{-4t}}}^{3y'} \overbrace{-4At - 4B - 4\cancel{Cte^{-4t}}}^{-4y} = 4t - 5e^{-4t}$$

$$(-4A)t + (3A - 4B) + (-5C)e^{-4t} = 4t - 5e^{-4t}$$

$-4A = 4 \Rightarrow A = -1$

$3A - 4B = 0 \Rightarrow -4B = 3 \Rightarrow B = -3/4$

$-5C = -5 \Rightarrow C = 1$

$$\boxed{y(t) = c_1 e^{-4t} + c_2 e^{t} - t - 3/4 + te^{-4t}}$$